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Schwarz-type problem of nonhomogeneous Cauchy–Riemann equation on a triangle ☆

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ABSTRACT

We consider the Schwarz-type boundary-value problem (BVP) of the nonhomogeneous Cauchy–Riemann equation on an isosceles orthogonal triangle. By the technique of plane parqueting and the Cauchy–Pompeiu formula on the triangle, the Schwarz–Poisson formula is obtained. We also investigate boundary behaviors of the Schwarz-type operator and the Pompeiu-type operator. Especially, boundary-values at the corners are proved to exist. Finally, the solution of the Schwarz-type BVP is explicitly obtained.

1. Introduction

A variety of BVPs of partial differential equations (PDEs) in two dimensions have been investigated in details, see for example [1–19]. The solutions of BVPs on some special domains are explicitly obtained. Those special domains include the unit disc, the half-plane, the half disc, the circular ring and so on. Especially, the convex polygon domain is considered in [16], and the boundary behavior at corner is unsolved in the general case.

Generally speaking, the theory of BVPs for analytic functions is closely connected with the theory of singular integral equation, index theory and other theories, and it also has many applications in shell theory, fluid dynamics, elasticity theory and so on [17,18]. Besides, Riemann–Hilbert technique can be used to solve BVPs of linear and of integrable nonlinear PDEs, and a general overview is presented in an excellent monograph [16]. Schwarz-type BVP of Cauchy–Riemann equation is a basic one, which has some influence on Dirichlet-type and Newmann-type BVP. Furthermore, the solutions of BVPs of second order complex equations are generally obtained by iterating the corresponding solutions of Cauchy–Riemann equation [15], and the well-known equations of second order are defined by the Laplace operator $\partial_z\partial_{\bar z}$ and the Bitsadze operator $\partial_{\bar z}^2$. The Laplace operator is just the simplest elliptic operator.

Now we introduce some notations used in the sequel.

The triangle with three vertices $0, 1, i$ is an isosceles orthogonal one, denoted as Δ. The boundary $\partial\Delta$ of the domain Δ consists of three sides, oriented counter-clockwise. The oriented segment from 1 to i, denoted as $[1, i]$, is parameterized by

$$\gamma_1 : t \mapsto 1 - t + it, \quad t \in [0, 1]. \tag{1.1}$$

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Similarly, the oriented segment from i to 0, say $[i, 0]$, is parameterized by $\gamma_2 : t \mapsto i(1-t)$, $t \in [0, 1]$, and the oriented segment from 0 to 1, $[0, 1]$, is parameterized by the identity mapping $\gamma_3 : t \mapsto t$, $t \in [0, 1]$. The three oriented sides $[1, i], [i, 0], [0, 1]$ are sometimes called $\gamma_1, \gamma_2, \gamma_3$, respectively here. Therefore $\partial\Delta = [1, i] \cup [i, 0] \cup [0, 1]$.

For two arbitrary points $\zeta_1, \zeta_2 \in [1, i]$, $\zeta_1 \neq \zeta_2$, if the oriented segment $[\zeta_1, \zeta_2]$ has the same orientation as $[1, i]$, we say $\zeta_1 \prec \zeta_2$. The relation $\prec$ is a linear order on $[1, i]$, and linear orders on $[i, 0]$ and $[0, 1]$ are similarly defined.

The Cauchy–Pompeiu formula is still valid for the triangle domain [3,15,19].

Theorem 1.1. *Any $w \in C^1(\Delta; \mathbb{C}) \cap C(\overline{\Delta}; \mathbb{C})$ can be represented by the formula*

$$w(z) = \frac{1}{2\pi i} \int\limits_{\partial\Delta} w(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int\limits_{\Delta} (\partial_{\bar{\zeta}} w)(\zeta) \frac{d\xi\, d\eta}{\zeta - z}, \quad z \in \Delta \tag{1.2}$$

with $\zeta = \xi + i\eta$, $\xi, \eta \in \mathbb{R}$, and the following equality is valid

$$\frac{1}{2\pi i} \int\limits_{\partial\Delta} w(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int\limits_{\Delta} (\partial_{\bar{\zeta}} w)(\zeta) \frac{d\xi\, d\eta}{\zeta - z} = 0, \quad z \in \mathbb{C} \setminus \overline{\Delta}. \tag{1.3}$$

In general, the explicit Schwarz–Poisson formula on the triangle domain Δ cannot be directly obtained from the classical Schwarz–Poisson formula on the unit disc or the half-plane by the conformal mapping, which can be expressed by the Schwarz–Christoffel formula.

In the present article, the Schwarz–Poisson formula on the triangle domain Δ is explicitly obtained by the technique of plane parqueting and the Cauchy–Pompeiu formulas (1.2) and (1.3), and then boundary behaviors of the Schwarz-type operator and the Pompeiu-type operator are investigated. In fact, plane parqueting, used in [1], is a triangulation of the complex plane $\mathbb{C}$. Finally, the Schwarz-type problem for the nonhomogeneous Cauchy–Riemann equation is studied, and the expression of solution is explicitly obtained.

In [21], the unknown Newmann boundary-values are explicitly expressed in terms of the given Dirichlet datum by the Fourier series, and conversely. In the present paper, the analytic solutions are directly expressed by the integrals of the Schwarz-type boundary-values, which can easily lead to the conclusion that the Dirichlet-type boundary-values can be expressed in terms of the Schwarz-type datum via the different series. In addition, the theory here can be similarly generalized to the arbitrary triangle domain, and the corresponding expressions of solutions are more complicated. Unfortunately, the method used here couldn't apply to the region outside a triangle, because it depends on the boundedness of the region. However, the results here can be generalized to the case of the higher order equations, such as polyanalytic equation [22].

2. Schwarz–Poisson formula on the triangle

In this section, by the technique of plane parqueting, the complex plane $\mathbb{C}$ is divided into infinitely many triangles, which are congruent to the triangle Δ.

Firstly, reflecting $z \in \Delta$ at the sloped edge of the triangle, $[1, i]$, one gets its symmetric point

$$z_1 = 1 - i(\bar{z} - 1) = -i\bar{z} + 1 + i. \tag{2.1}$$

By (2.1), the triangle domain Δ is bijectively mapped onto the new triangle domain Δ^1 with the vertices $1, 1+i, i$, and the boundary of the triangle domain Δ is bijectively mapped onto the boundary of the triangle domain Δ^1. Secondly, reflecting at the segment from 1 to $1+i$, one easily gets the symmetric point of z_1 represented by

$$z_2 = 1 - i(z - 1) = -iz + 1 + i. \tag{2.2}$$

The triangle Δ^1 is bijectively transformed into the new triangle Δ^2 with the vertices $1, 2+i, 1+i$. Thirdly, the triangle Δ^2 is reflected into the new one Δ^3 with the vertices $1, 2, 2+i$, and the symmetric point of z_2 with respect to the side $[1, 2+i]$ is

$$z_3 = 2 - \bar{z}. \tag{2.3}$$

Continuing reflection at the segment $[2, 2+i]$, the symmetric point of z_3 is

$$z_4 = 2 + z, \tag{2.4}$$

which indicates that the reflection along the horizontal direction possesses the minimum positive period 2.

By the reflection along the vertical direction, the basic period $2i$ is similarly obtained. By the reflection on the real axis, the corresponding symmetric points of z, z_1, z_2, z_3 are denoted as $\bar{z}, \overline{z_1}, \overline{z_2}, \overline{z_3}$, respectively, and the triangles $\Delta, \Delta^1, \Delta^2, \Delta^3$ are bijectively mapped into the new triangles $\widetilde{\Delta}, \widetilde{\Delta^1}, \widetilde{\Delta^2}, \widetilde{\Delta^3}$, respectively. Those eight triangles are put together into a basic square, denoted by $\square$. The reflection along the horizontal and vertical direction is equivalent to extending the basic square $\square$ by the basic double periods $2, 2i$. Let

$$\Omega_{m,n} = 2m + 2ni, \quad m, n \in \mathbb{Z} \tag{2.5}$$

and define the (m,n)-square

$$\square_{m,n} = \{z + 2m + 2ni \colon z \in \square\}, \tag{2.6}$$

one has $\square = \square_{0,0}$ and $\mathbb{C} = \bigcup_{m,n=-\infty}^{+\infty} \overline{\square}_{m,n}$.

By (2.6), the relationship between the (m,n)-square $\square_{m,n}$ and the basic square $\square$ can be described as $\square_{m,n} = \square + \Omega_{m,n}$. In the same manners, we introduce some other symbols $\Delta_{m,n}$, $\widetilde{\Delta}_{m,n}$ and $\Delta^k_{m,n}$, $\widetilde{\Delta^k}_{m,n}$, $k = 1, 2, 3$. For example, $\Delta_{m,n} = \Delta + \Omega_{m,n}$, where $\Omega_{m,n}$ is defined in (2.5).

Applying Theorem 1.1 to triangle domains $\Delta_{m,n}$, $\Delta^2_{m,n}$, $\widetilde{\Delta}^1_{m,n}$ and $\widetilde{\Delta}^3_{m,n}$, respectively, one easily gets four equalities

$$w(z)\delta_{n0}\delta_{m0} = \frac{1}{2\pi i}\int_{\partial\Delta} w(\zeta)\frac{d\zeta}{\zeta - z - 2m - 2ni} - \frac{1}{\pi}\int_{\Delta} (\partial_{\bar\zeta} w)(\zeta)\frac{d\xi\, d\eta}{\zeta - z - 2m - 2ni}, \quad z \in \Delta, \tag{2.7}$$

where δ_{n0}, δ_{m0} are Kronecker's symbols,

$$0 = \frac{1}{2\pi i}\int_{\partial\Delta} w(\zeta)\frac{d\zeta}{\zeta + iz - (2m+1) - (2n+1)i} - \frac{1}{\pi}\int_{\Delta} (\partial_{\bar\zeta} w)(\zeta)\frac{d\xi\, d\eta}{\zeta + iz - (2m+1) - (2n+1)i}, \quad z \in \Delta, \tag{2.8}$$

$$0 = \frac{1}{2\pi i}\int_{\partial\Delta} w(\zeta)\frac{d\zeta}{\zeta - iz - (2m+1) - (2n-1)i} - \frac{1}{\pi}\int_{\Delta} (\partial_{\bar\zeta} w)(\zeta)\frac{d\xi\, d\eta}{\zeta - iz - (2m+1) - (2n-1)i}, \quad z \in \Delta \tag{2.9}$$

and

$$0 = \frac{1}{2\pi i}\int_{\partial\Delta} w(\zeta)\frac{d\zeta}{\zeta + z - (2m+2) - 2ni} - \frac{1}{\pi}\int_{\Delta} (\partial_{\bar\zeta} w)(\zeta)\frac{d\xi\, d\eta}{\zeta + z - (2m+2) - 2ni}, \quad z \in \Delta. \tag{2.10}$$

Also by Theorem 1.1,

$$0 = \frac{1}{2\pi i}\int_{\partial\Delta} w(\zeta)\frac{d\zeta}{\zeta - (\lambda + \Omega_{m,n})} - \frac{1}{\pi}\int_{\Delta} (\partial_{\bar\zeta} w)(\zeta)\frac{d\xi\, d\eta}{\zeta - (\lambda + \Omega_{m,n})}, \quad \lambda = z_1, z_3, \bar z, \overline{z_2}. \tag{2.11}$$

When $\lambda = z_1$, taking the complex conjugation on both sides of (2.11) gives

$$\begin{aligned} 0 = &-\frac{1}{2\pi i}\int_{[1,i]} \overline{w(\zeta)}\frac{d\zeta}{\zeta - z + 2n + 2mi} - \frac{1}{2\pi i}\int_{[i,0]} \overline{w(\zeta)}\frac{d\zeta}{\zeta + iz + (2m+1) - (2n+1)i} \\ &- \frac{1}{2\pi i}\int_0^1 \overline{w(x)}\frac{dx}{x - iz - (2m+1) + (2n+1)i} \\ &- \frac{1}{\pi}\int_{\Delta} \overline{(\partial_{\bar\zeta} w)(\zeta)}\frac{d\xi\, d\eta}{\bar\zeta - iz - (2m+1) + (2n+1)i}, \quad z \in \Delta, \end{aligned} \tag{2.12}$$

since

$$\zeta = \begin{cases} -i\bar\zeta + i + 1, & \zeta \in [1,i], \\ -\bar\zeta, & \zeta \in [i,0], \\ \bar\zeta, & \zeta \in [0,1]. \end{cases} \tag{2.13}$$

Similarly, when $\lambda = z_3, \bar z, \bar z_2$, (2.11) is equivalently transformed to

$$\begin{aligned} 0 = &-\frac{1}{2\pi i}\int_{[1,i]} \overline{w(\zeta)}\frac{d\zeta}{\zeta - iz + (2n-1) + (2m+1)i} - \frac{1}{2\pi i}\int_{[i,0]} \overline{w(\zeta)}\frac{d\zeta}{\zeta - z + (2m+2) - 2ni} \\ &- \frac{1}{2\pi i}\int_0^1 \overline{w(x)}\frac{dx}{x + z - (2m+2) + 2ni} - \frac{1}{\pi}\int_{\Delta} \overline{(\partial_{\bar\zeta} w)(\zeta)}\frac{d\xi\, d\eta}{\bar\zeta + z - (2m+2) + 2ni}, \quad z \in \Delta, \end{aligned} \tag{2.14}$$

$$0 = -\frac{1}{2\pi i}\int\limits_{[1,i]} \overline{w(\zeta)}\frac{d\zeta}{\zeta + iz + (2n-1) + (2m-1)i} - \frac{1}{2\pi i}\int\limits_{[i,0]} \overline{w(\zeta)}\frac{d\zeta}{\zeta + z + 2m - 2ni}$$
$$- \frac{1}{2\pi i}\int\limits_0^1 \overline{w(x)}\frac{dx}{x - z - 2m + 2ni} - \frac{1}{\pi}\int\limits_{\Delta} \overline{(\partial_{\bar\zeta} w)(\zeta)}\frac{d\xi\, d\eta}{\bar\zeta - z - 2m + 2ni}, \quad z \in \Delta, \tag{2.15}$$

and

$$0 = -\frac{1}{2\pi i}\int\limits_{[1,i]} \overline{w(\zeta)}\frac{d\zeta}{\zeta + z + (2n-2) + 2mi} - \frac{1}{2\pi i}\int\limits_{[i,0]} \overline{w(\zeta)}\frac{d\zeta}{\zeta - iz + (2m+1) - (2n-1)i}$$
$$- \frac{1}{2\pi i}\int\limits_0^1 \overline{w(x)}\frac{dx}{x + iz - (2m+1) + (2n-1)i} - \frac{1}{\pi}\int\limits_{\Delta} \overline{(\partial_{\bar\zeta} w)(\zeta)}\frac{d\xi\, d\eta}{\bar\zeta + iz - (2m+1) + (2n-1)i}, \quad z \in \Delta. \tag{2.16}$$

For $M, N \in \mathbb{N}$, let $R_{M,N} = \{(k, j)\colon\ |k| \leqslant M,\ |j| \leqslant N,\ k, j \in \mathbb{Z}\}$ be the finite set of double indices. If the limit $\lim_{(M,N)\to(\infty,\infty)} \sum_{(m,n)\in R_{M,N}} f_{m,n}(\zeta, z)$ exists, then the double series $\sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}} f_{m,n}(\zeta, z)$, simply $\sum_{m,n} f_{m,n}(\zeta, z)$, is convergent along the rectangles with center at the origin, written as

$$\sum_{m,n} f_{m,n}(\zeta, z) = \lim_{(M,N)\to(\infty,\infty)} \sum_{(m,n)\in R_{M,N}} f_{m,n}(\zeta, z). \tag{2.17}$$

If the limit $\lim_{N\to\infty}\lim_{M\to\infty} \sum_{(m,n)\in R_{M,N}} f_{m,n}(\zeta, z)$ exists, we denote by

$$\sum_n\sum_m f_{m,n}(\zeta, z) = \lim_{N\to\infty}\lim_{M\to\infty} \sum_{(m,n)\in R_{M,N}} f_{m,n}(\zeta, z). \tag{2.18}$$

Suppose that $E \subset \mathbb{C}$ be a set. Some associated sets of E are respectively defined by

$$E^1 = -E, \qquad E^2 = -iE + 1 + i, \qquad E^3 = iE + 1 - i, \tag{2.19}$$

and

$$E^k_{m,n} = E^k + \Omega_{m,n}, \quad k = 0, 1, 2, 3, \tag{2.20}$$

where $E^0 = E$ and $\Omega_{m,n}$ is given in (2.5). Let $\underline{E}$ be the reflection set of E on the real axis, i.e., $\underline{E} = \{\bar z\colon\ z \in E\}$. Furthermore, $\underline{E}^k$, $k = 1, 2, 3$, and $\underline{E}^k_{m,n}$ are defined by (2.19) and (2.20), respectively.

Lemma 2.1. *Suppose that $S, E, W \subset \mathbb{C}$ be three bounded sets. If $S \cap E_{m,n} = \emptyset$, $S \cap W_{m,n} = \emptyset$ for all $(m, n) \in \mathbb{Z}\times\mathbb{Z}$, then the double series*

$$\sum_{m,n}\left[\frac{1}{(\zeta - z) - \Omega_{m,n}} - \frac{1}{(\zeta - w) - \Omega_{m,n}}\right] \tag{2.21}$$

is uniformly convergent with respect to $(\zeta, z, w) \in S \times E \times W$ in the sense of (2.17)*, where $E_{m,n} = E^0_{m,n}$ and $W_{m,n} = W^0_{m,n}$ are defined by* (2.20)*.*

Proof. Since $S \cap E_{m,n} = \emptyset$ and $S \cap W_{m,n} = \emptyset$ for all $(m, n) \in \mathbb{Z}\times\mathbb{Z}$, every term in (2.21) is meaningful when $(\zeta, z, w) \in S \times E \times W$. Observe

$$\sum_{(m,n)\in R_{M,N}} \frac{1}{(\zeta - z) - (2m + 2ni)} = -\frac{1}{\zeta - z} + \sum_{m=-M}^{M} \frac{1}{(\zeta - z) - 2m} + \sum_{n=-N}^{N} \frac{1}{(\zeta - z) - 2ni}$$
$$+ 4(\zeta - z)\sum_{(m,n)=(1,1)}^{(M,N)} \frac{(\zeta - z)^2 - 4m^2 + 4n^2}{[(\zeta - z)^2 - 4m^2 + 4n^2]^2 + 64m^2n^2} \tag{2.22}$$

and

$$
\begin{aligned}
r_{m,n}(\zeta, z, w) &= \frac{(\zeta - z)^2 - 4m^2 + 4n^2}{[(\zeta - z)^2 - 4m^2 + 4n^2]^2 + 64m^2n^2} - \frac{(\zeta - w)^2 - 4m^2 + 4n^2}{[(\zeta - w)^2 - 4m^2 + 4n^2]^2 + 64m^2n^2} \\
&= \frac{\{[(\zeta - z)^2 - 4m^2 + 4n^2] \cdot [(\zeta - w)^2 - 4m^2 + 4n^2] - 64m^2n^2\} \cdot [(\zeta - w)^2 - (\zeta - z)^2]}{\{[(\zeta - z)^2 - 4m^2 + 4n^2]^2 + 64m^2n^2\} \cdot \{[(\zeta - w)^2 - 4m^2 + 4n^2]^2 + 64m^2n^2\}}.
\end{aligned} \tag{2.23}
$$

By computation, there exist $M, N > 0$ such that $m > M$ and $n > N$ imply the estimates

$$
\begin{cases}
\left|\left[(\zeta - z)^2 - 4m^2 + 4n^2\right]^2 + 64m^2n^2\right| \geqslant 16\left(m^2 + n^2\right)^2 - d_1\left(d_1 + 8\left(m^2 + n^2\right)\right) > 0, \\
\left|\left[(\zeta - w)^2 - 4m^2 + 4n^2\right]^2 + 64m^2n^2\right| \geqslant 16\left(m^2 + n^2\right)^2 - d_2\left(d_2 + 8\left(m^2 + n^2\right)\right) > 0, \\
\left|\left\{\left[(\zeta - z)^2 - 4m^2 + 4n^2\right] \cdot \left[(\zeta - w)^2 - 4m^2 + 4n^2\right] - 64m^2n^2\right\} \cdot \left[(\zeta - w)^2 - (\zeta - z)^2\right]\right| \\
\quad \leqslant (d_1 + d_2) \cdot \left[16\left(m^2 + n^2\right)^2 + 4\left(m^2 + n^2\right)(d_1 + d_2) + d_1 d_2\right]
\end{cases}
$$

with

$$
\begin{cases}
d_1 = \sup\left\{|\zeta - z|^2 \colon (\zeta, z) \in S \times E\right\}, \\
d_2 = \sup\left\{|\zeta - w|^2 \colon (\zeta, w) \in S \times W\right\}.
\end{cases}
$$

Since S, E, W are bounded sets, $d_1 < +\infty$ and $d_2 < +\infty$. Thus, when $m > M$, $n > N$,

$$
\left|r_{m,n}(\zeta, z, w)\right| \leqslant \frac{2d \cdot [16(m^2 + n^2)^2 + 8(m^2 + n^2)d + d^2]}{[16(m^2 + n^2)^2 - 8(m^2 + n^2)d - d^2]^2}
$$

with $d = \max\{d_1, d_2\}$. By Lemma 1.5 in [20] on p. 268, the double series

$$
\sum_{(m,n)=(M,N)}^{(\infty,\infty)} \frac{2d \cdot [16(m^2 + n^2)^2 + 8(m^2 + n^2)d + d^2]}{[16(m^2 + n^2)^2 - 8(m^2 + n^2)d - d^2]^2}
$$

converges, and hence, by (2.22) and (2.23), the double series (2.21) is uniformly convergent with respect to $(\zeta, z, w) \in S \times E \times W$ in the sense of (2.17). $\square$

Define four functions as follows

$$
\begin{aligned}
g_{m,n}(\zeta, z) &= \frac{1}{\zeta - z - 2m - 2ni} + \frac{1}{\zeta + iz - (2m + 1) - (2n + 1)i} \\
&\quad + \frac{1}{\zeta - iz - (2m + 1) - (2n - 1)i} + \frac{1}{\zeta + z - (2m + 2) - 2ni}
\end{aligned} \tag{2.24}
$$

and

$$
\begin{aligned}
h^1_{m,n}(\zeta, z) &= \frac{1}{\zeta - z + 2n + 2mi} + \frac{1}{\zeta - iz + (2n - 1) + (2m + 1)i} \\
&\quad + \frac{1}{\zeta + iz + (2n - 1) + (2m - 1)i} + \frac{1}{\zeta + z + (2n - 2) + 2mi};
\end{aligned} \tag{2.25}
$$

$$
\begin{aligned}
h^2_{m,n}(\zeta, z) &= \frac{1}{\zeta + iz + (2m + 1) - (2n + 1)i} + \frac{1}{\zeta - z + (2m + 2) - 2ni} \\
&\quad + \frac{1}{\zeta + z + 2m - 2ni} + \frac{1}{\zeta - iz + (2m + 1) - (2n - 1)i};
\end{aligned} \tag{2.26}
$$

$$
\begin{aligned}
h^3_{m,n}(\zeta, z) &= \frac{1}{\zeta - iz - (2m + 1) + (2n + 1)i} + \frac{1}{\zeta + z - (2m + 2) + 2ni} \\
&\quad + \frac{1}{\zeta - z - 2m + 2ni} + \frac{1}{\zeta + iz - (2m + 1) + (2n - 1)i}.
\end{aligned} \tag{2.27}
$$

One easily gets

$$
g_{m,n}(\zeta, z) = \begin{cases}
h^1_{-n,-m}(\zeta, z), \\
h^2_{-(m+1),n}(\zeta, z), \\
h^3_{m,-n}(\zeta, z).
\end{cases} \tag{2.28}
$$

For $M, N \in \mathbb{N}$, define

$$G_{M,N}(\zeta,z)=\sum_{(m,n)\in R_{M,N}} g_{m,n}(\zeta,z) \tag{2.29}$$

and

$$H^k_{M,N}(\zeta,z)=\sum_{(m,n)\in R_{M,N}} h^k_{m,n}(\zeta,z),\quad k=1,2,3. \tag{2.30}$$

Lemma 2.2. *Suppose that $S, E\subset\mathbb{C}$ be two bounded sets. If $S\cap E^{\ell}_{m,n}=\emptyset$ for all $m,n\in\mathbb{Z}$ and $\ell=0,1,2,3$, then*

$$\sum_{m,n}\big[g_{m,n}(\zeta,z)-h^k_{m,n}(\zeta,z)\big]=0,\quad \zeta\in S,\ z\in E,\ k=1,3$$

and

$$\sum_n\sum_m\big[g_{m,n}(\zeta,z)-h^2_{m,n}(\zeta,z)\big]=0,\quad \zeta\in S,\ z\in E,$$

where $g_{m,n}$ and $h^k_{m,n}$, $k=1,2,3$, are defined by (2.24)–(2.27)*, respectively, and $E^{\ell}_{m,n}$ is defined by* (2.20)*.*

Proof. If $S\cap E^{\ell}_{m,n}=\emptyset$ for all $m,n\in\mathbb{Z}$ and $\ell=0,1,2,3$, then $g_{m,n}(\zeta,z)-h^k_{m,n}(\zeta,z)$ is meaningful for all $m,n\in\mathbb{Z}$ and $k=1,2,3$ when $(\zeta,z)\in S\times E$. By (2.28),

$$G_{M,N}(\zeta,z)=H^1_{M,N}(\zeta,z)=H^3_{M,N}(\zeta,z)=H^2_{M,N}(\zeta,z)+\sum_{n=-N}^{N}\big[h^2_{-(M+1),n}(\zeta,z)-h^2_{M,n}(\zeta,z)\big], \tag{2.31}$$

where $G_{M,N}$ and $H^k_{M,N}$, $k=1,2,3$, are defined by (2.29) and (2.30), respectively. The desired conclusion follows from (2.31). □

Lemma 2.3. *Suppose that $S, E\subset\mathbb{C}$ be two bounded sets. If $S\cap E^{\ell}_{m,n}=\emptyset$ for all $m,n\in\mathbb{Z}$ and $\ell=0,1,2,3$, then the double series*

$$\sum_{m,n}\big[g_{m,n}(\zeta,z)-g_{m,n}(\zeta,\alpha)\big],\qquad \sum_{m,n}\big[h^k_{m,n}(\zeta,z)-h^k_{m,n}(\zeta,\alpha)\big],\quad k=1,2,3,$$

are uniformly convergent with respect to $(\zeta,z)\in S\times E$ in the sense of (2.17)*, where α is a fixed constant satisfying*

$$S\cap\big\{\alpha+\Omega_{m,n},-\alpha+\Omega_{m,n},(-i\alpha+1+i)+\Omega_{m,n},(i\alpha+1-i)+\Omega_{m,n}\colon m,n\in\mathbb{Z}\big\}=\emptyset, \tag{2.32}$$

$\Omega_{m,n}$, $E^{\ell}_{m,n}$ are given by (2.5), (2.20)*, respectively. Moreover,*

$$\sum_{m,n}\big[g_{m,n}(\zeta,z)-g_{m,n}(\zeta,\alpha)\big]=\sum_{m,n}\big[h^k_{m,n}(\zeta,z)-h^k_{m,n}(\zeta,\alpha)\big],\quad k=1,2,3. \tag{2.33}$$

Proof. $S\cap E^{\ell}_{m,n}=\emptyset$ for all $m,n\in\mathbb{Z}$ and $\ell=0,1,2,3$ and the condition (2.32) guarantees that the terms

$$g_{m,n}(\zeta,z)-g_{m,n}(\zeta,\alpha),\qquad h^k_{m,n}(\zeta,z)-h^k_{m,n}(\zeta,\alpha),\quad k=1,2,3,$$

are meaningful for all $m,n\in\mathbb{Z}$ when $(\zeta,z)\in S\times E$. The former part of the lemma follows from Lemma 2.1. The relation (2.31) directly leads to (2.33) by Lemma 2.2. □

The following result directly follows from Lemma 2.3.

Corollary 2.1. *If $(\zeta,z)\in[i,0]\times(\overline{\Delta}\setminus[i,0])$, then*

$$\sum_{m,n}\big[g_{m,n}(\zeta,z)-g_{m,n}(\zeta,\bar z)\big]=\sum_{m,n}\big[h^2_{m,n}(\zeta,z)-h^2_{m,n}(\zeta,\bar z)\big],$$

where $g_{m,n}$, $h^2_{m,n}$ by (2.24) *and* (2.26)*, respectively.*

By Lemma 2.1 and (2.31), one has the following.

Lemma 2.4. *Suppose that $S, E \subset \mathbb{C}$ be two bounded sets. If $S \cap E^{\ell}_{m,n} = \emptyset$, $S \cap \underline{E}^{\ell}_{m,n} = \emptyset$ for all $m, n \in \mathbb{Z}$ and $\ell = 0, 1, 2, 3$, then the double series*

$$\sum_{m,n}\big[g_{m,n}(\zeta, z) - h^k_{m,n}(\zeta, \bar{z})\big] = \sum_{m,n}\big[g_{m,n}(\zeta, z) - g_{m,n}(\zeta, \bar{z})\big], \quad k = 1, 3,$$

are uniformly convergent with respect to $(\zeta, z) \in S \times E$ in the sense of (2.17), where $\underline{E} = \{\bar{z}\colon z \in E\}$.

Now adding (2.7), (2.8), (2.9) and (2.10), and taking the sum for all the indices $(m, n) \in R_{M,N}$, we have

$$w(z) = \frac{1}{2\pi i}\int_{\partial\Delta} w(\zeta)G_{M,N}(\zeta, z)\,d\zeta - \frac{1}{\pi}\int_{\Delta}(\partial_{\bar\zeta} w)(\zeta)G_{M,N}(\zeta, z)\,d\xi\,d\eta, \quad z \in \Delta, \tag{2.34}$$

which in turn is equivalent to

$$\begin{aligned} w(z) = w(\alpha) &+ \frac{1}{2\pi i}\int_{\partial\Delta} w(\zeta)\big[G_{M,N}(\zeta, z) - G_{M,N}(\zeta, \alpha)\big]\,d\zeta \\ &- \frac{1}{\pi}\int_{\Delta}(\partial_{\bar\zeta} w)(\zeta)\big[G_{M,N}(\zeta, z) - G_{M,N}(\zeta, \alpha)\big]\,d\xi\,d\eta, \quad z \in \Delta, \end{aligned} \tag{2.35}$$

where $\alpha \in \Delta$ is a fixed constant and $G_{M,N}$ is given in (2.29). Similarly, from (2.12), (2.14), (2.15) and (2.16), one has

$$\begin{aligned} &-\frac{1}{2\pi i}\int_{[1,i]} \overline{w(\zeta)}H^1_{M,N}(\zeta, z)\,d\zeta - \frac{1}{2\pi i}\int_{[i,0]} \overline{w(\zeta)}H^2_{M,N}(\zeta, z)\,d\zeta - \frac{1}{2\pi i}\int_0^1 \overline{w(x)}H^3_{M,N}(x, z)\,dx \\ &\quad - \frac{1}{\pi}\int_{\Delta}\overline{(\partial_{\bar\zeta} w)(\zeta)}H^3_{M,N}(\bar\zeta, z)\,d\xi\,d\eta = 0, \quad z \in \Delta, \end{aligned} \tag{2.36}$$

which is similarly equivalent to

$$\begin{aligned} 0 = &-\frac{1}{2\pi i}\int_{[1,i]} \overline{w(\zeta)}\big[H^1_{M,N}(\zeta, z) - H^1_{M,N}(\zeta, \alpha)\big]\,d\zeta - \frac{1}{2\pi i}\int_{[i,0]} \overline{w(\zeta)}\big[H^2_{M,N}(\zeta, z) - H^2_{M,N}(\zeta, \alpha)\big]\,d\zeta \\ &- \frac{1}{2\pi i}\int_0^1 \overline{w(x)}\big[H^3_{M,N}(x, z) - H^3_{M,N}(x, \alpha)\big]\,dx - \frac{1}{\pi}\int_{\Delta}\overline{(\partial_{\bar\zeta} w)(\zeta)}\big[H^3_{M,N}(\bar\zeta, z) - H^3_{M,N}(\bar\zeta, \alpha)\big]\,d\xi\,d\eta, \quad z \in \Delta, \end{aligned} \tag{2.37}$$

where $\alpha \in \Delta$ is a fixed constant and $H^k_{M,N}$ is given by (2.30). Now the Schwarz–Poisson formula can be explicitly expressed for the triangle domain Δ.

Theorem 2.1. *If $w \in C^1(\Delta; \mathbb{C}) \cap C(\overline{\Delta}; \mathbb{C})$, then*

$$\begin{aligned} w(z) = w(\alpha) &+ \frac{1}{\pi i}\int_{\partial\Delta} \operatorname{Re} w(\zeta)\sum_{m,n}\big[g_{m,n}(\zeta, z) - g_{m,n}(\zeta, \alpha)\big]\,d\zeta \\ &- \frac{1}{\pi}\int_{\Delta}\Big\{(\partial_{\bar\zeta} w)(\zeta)\sum_{m,n}\big[g_{m,n}(\zeta, z) - g_{m,n}(\zeta, \alpha)\big] - \overline{(\partial_{\bar\zeta} w)(\zeta)}\sum_{m,n}\big[g_{m,n}(\bar\zeta, z) - g_{m,n}(\bar\zeta, \alpha)\big]\Big\}\,d\xi\,d\eta, \quad z \in \Delta, \end{aligned} \tag{2.38}$$

where $\alpha \in \Delta$ is a fixed constant.

Proof. Subtracting (2.37) from (2.35), by (2.31), one easily gets

$$\begin{aligned} w(z) = w(\alpha) &+ \frac{1}{2\pi i}\int_{[1,i]} \operatorname{Re} w(\zeta)\big[G_{M,N}(\zeta, z) + H^1_{M,N}(\zeta, z) - G_{M,N}(\zeta, \alpha) - H^1_{M,N}(\zeta, \alpha)\big]\,d\zeta \\ &+ \frac{1}{2\pi i}\int_{[i,0]} \operatorname{Re} w(\zeta)\big[G_{M,N}(\zeta, z) + H^2_{M,N}(\zeta, z) - G_{M,N}(\zeta, \alpha) - H^2_{M,N}(\zeta, \alpha)\big]\,d\zeta \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2\pi i}\int\limits_{[i,0]} i\operatorname{Im} w(\zeta)\big\{\big[G_{M,N}(\zeta,z)-G_{M,N}(\zeta,\alpha)\big]-\big[H^2_{M,N}(\zeta,z)-H^2_{M,N}(\zeta,\alpha)\big]\big\}\,d\zeta\\
&+\frac{1}{2\pi i}\int\limits_0^1 \operatorname{Re} w(x)\big[G_{M,N}(x,z)+H^3_{M,N}(x,z)-G_{M,N}(x,\alpha)-H^3_{M,N}(x,\alpha)\big]\,dx\\
&-\frac{1}{\pi}\int\limits_{\Delta}\big\{(\partial_{\bar\zeta} w)(\zeta)\big[G_{M,N}(\zeta,z)-G_{M,N}(\zeta,\alpha)\big]-\overline{(\partial_{\bar\zeta} w)(\zeta)}\big[H^3_{M,N}(\bar\zeta,z)-H^3_{M,N}(\bar\zeta,\alpha)\big]\big\}\,d\xi\,d\eta,\quad z\in\Delta,
\end{aligned}
\tag{2.39}
$$

where $\alpha\in\Delta$ is a fixed constant. By Lemma 2.3 and (2.31), passing to the limit, (2.39) leads to the desired conclusion (2.38). □

3. Boundary behavior of the Schwarz-type operator

Firstly, we introduce the Schwarz-type operator S_α on the boundary of the triangle domain Δ as follows

$$
S_\alpha[\rho](z)=\frac{1}{\pi i}\int\limits_{\partial\Delta}\rho(\zeta)\sum_{m,n}\left[g_{m,n}(\zeta,z)-\frac{g_{m,n}(\zeta,\alpha)+g_{m,n}(\zeta,\bar\alpha)}{2}\right]d\zeta,\quad z\in\Delta,
\tag{3.1}
$$

where $\alpha\in\Delta$ is a fixed constant, $\rho\in C(\partial\Delta;\mathbb{C})$ and $g_{m,n}(\zeta,z)$ is defined by (2.24). By Lemma 2.3, the double series in (3.1) is uniformly convergent with respect to $(\zeta,z)\in\partial\Delta\times E$ with $E\subset\Delta$. Obviously

$$
S_\alpha[\rho](\alpha)=\frac{1}{2\pi i}\int\limits_{\partial\Delta}\rho(\zeta)\sum_{m,n}\big[g_{m,n}(\zeta,\alpha)-g_{m,n}(\zeta,\bar\alpha)\big]\,d\zeta.
\tag{3.2}
$$

Theorem 3.1. *If $\rho\in C(\partial\Delta;\mathbb{C})$, then $S_\alpha[\rho](z)$ defined by* (3.1) *is analytic in the triangle domain Δ, denoted as $S_\alpha[\rho]\in A(\Delta)$.*

Proof. Let

$$
F_{m,n}(z)=\frac{1}{\pi i}\int\limits_{\partial\Delta}\rho(\zeta)\left[g_{m,n}(\zeta,z)-\frac{g_{m,n}(\zeta,\alpha)+g_{m,n}(\zeta,\bar\alpha)}{2}\right]d\zeta,\quad z\in\Delta.
\tag{3.3}
$$

Obviously $F_{m,n}\in A(\Delta)$. By Lemma 2.3 and the Wereistrass' theorem for complex analytic sequence, the desired result is obtained. □

The classical Schwarz kernel for the upper half-plane $\mathbb{C}^+$ is

$$
\frac{z-\bar z}{|x-z|^2},\quad (x,z)\in(-\infty,+\infty)\times\mathbb{C}^+
\tag{3.4}
$$

and one has the classical boundary behavior

$$
\lim_{z\in\mathbb{C}^+,\,z\to t}\frac{1}{2\pi i}\int\limits_a^b\rho(x)\frac{z-\bar z}{|x-z|^2}\,dx=\rho(t),\quad t\in(a,b)\subset(-\infty,+\infty)
\tag{3.5}
$$

with $\rho\in C([a,b];\mathbb{C})$, $a<b$.

Let $\mathbb{C}_1^+$ denote the right half-plane, i.e., $\mathbb{C}_1^+=\{z=x+iy\colon\ x>0,\ x,y\in\mathbb{R}\}$. By the conformal mapping $\zeta=iz$ between $\mathbb{C}_1^+$ and $\mathbb{C}^+$, and (3.4), the Schwarz kernel of the right half-plane is $-\frac{z+\bar z}{|\zeta-z|^2}$, $(\zeta,z)\in(+\infty i,-\infty i)\times\mathbb{C}_1^+$, and

$$
\lim_{z\in\mathbb{C}_1^+,\,z\to t}\frac{-1}{2\pi i}\int\limits_{[ai,bi]}\rho(\zeta)\frac{z+\bar z}{|\zeta-z|^2}\,d\zeta=\rho(t),\quad t\in(ai,bi)\subset(+\infty i,-\infty i)
\tag{3.6}
$$

with $\rho\in C([ai,bi];\mathbb{C})$, $ai\prec bi$.

Let L: $\zeta=1+t(i-1)$, $-\infty<t<+\infty$ be the straight line through two points 1 and i on the complex plane $\mathbb{C}$, which possesses the same positive orientation as $[1,i]$. L divides the complex plane $\mathbb{C}$ into two half-planes. The half-plane including the origin is denoted as $\mathbb{C}_2^-$. By the conformal mapping $z=1+e^{-\frac{3}{4}\pi i}(\zeta-1)$, the half-plane $\mathbb{C}^+$ is mapped onto $\mathbb{C}_2^-$. Also by (3.4), the Schwarz kernel of the half-plane $\mathbb{C}_2^-$ is $\frac{i(z-1)-(\bar z-1)}{|\zeta-z|^2}$, $(\zeta,z)\in L\times\mathbb{C}_2^-$, and

$$\lim_{z\in\mathbb{C}_2^-,\, z\to t} \frac{1}{2\pi i} \int_{[\mu,\nu]} \rho(\zeta) \frac{i(z-1)-(\bar{z}-1)}{|\zeta - z|^2}\, d\zeta = \rho(t), \quad t \in (\mu,\nu) \subset L \tag{3.7}$$

with $\rho \in C([\mu,\nu];\mathbb{C})$, $\mu \prec \nu$.

Lemma 3.1. *If $\rho \in C([i,0];\mathbb{C})$ and $\rho(i)=\rho(0)=0$, then*

$$\lim_{z\in\Delta,\, z\to t} \frac{1}{2\pi i} \int_{[i,0]} \rho(\zeta) \frac{z+\bar{z}}{(\zeta - z)(\zeta + \bar{z})}\, d\zeta = \rho(t), \quad t \in [i,0]; \tag{3.8}$$

if $\rho \in C([1,i];\mathbb{C})$ and $\rho(i)=\rho(1)=0$, then

$$\lim_{z\in\Delta,\, z\to t} \frac{1}{2\pi i} \int_{[1,i]} \rho(\zeta) \frac{z+(i\bar{z}-1-i)}{(\zeta - z)[\zeta + (i\bar{z}-1-i)]}\, d\zeta = \rho(t), \quad t \in [1,i]; \tag{3.9}$$

if $\rho \in C([0,1];\mathbb{C})$ and $\rho(1)=\rho(0)=0$, then

$$\lim_{z\in\Delta,\, z\to t} \frac{1}{2\pi i} \int_0^1 \rho(x) \frac{z-\bar{z}}{(x - z)(x - \bar{z})}\, dx = \rho(t), \quad t \in [0,1]. \tag{3.10}$$

Proof. If $\rho \in C([i,0];\mathbb{C})$ and $\rho(i)=\rho(0)=0$, we define the function

$$\widetilde{\rho}(\zeta) = \begin{cases} \rho(\zeta), & \zeta \in [i,0], \\ 0, & \zeta \in (+\infty i, i) \cup (0, -\infty i), \end{cases} \tag{3.11}$$

and obviously $\widetilde{\rho} \in C((+\infty i, -\infty i);\mathbb{C})$. Hence (3.6) implies

$$\lim_{z\in\mathbb{C}_1^+,\, z\to t} \frac{-1}{2\pi i} \int_{[2i,-i]} \widetilde{\rho}(\zeta) \frac{z+\bar{z}}{|\zeta - z|^2}\, d\zeta = \widetilde{\rho}(t), \quad t \in [i,0] \subset (2i,-i)$$

which leads to

$$\lim_{z\in\mathbb{C}_1^+,\, z\to t} \frac{-1}{2\pi i} \int_{[i,0]} \rho(\zeta) \frac{z+\bar{z}}{|\zeta - z|^2}\, d\zeta = \rho(t), \quad t \in [i,0]$$

by the definition (3.11) of $\widetilde{\rho}$. Since

$$-\frac{z+\bar{z}}{|\zeta - z|^2} = \frac{z+\bar{z}}{(\zeta - z)(\zeta + \bar{z})}, \quad \zeta \in [i,0]$$

by (2.13), it follows that (3.8) is valid. Similarly, (3.9) and (3.10) remain true. □

Lemma 3.2. *If $\rho \in C([i,0];\mathbb{C})$ and $\rho(i)=\rho(0)=0$, then*

$$\lim_{z\in\Delta,\, z\to t} \frac{1}{2\pi i} \int_{[i,0]} \rho(\zeta) \sum_{m,n} \big[g_{m,n}(\zeta,z) - g_{m,n}(\zeta,\bar{z})\big]\, d\zeta = \rho(t), \quad t \in [i,0], \tag{3.12}$$

$$\lim_{z\in\Delta,\, z\to t} \frac{1}{2\pi i} \int_{[i,0]} \rho(\zeta) \sum_{m,n} \big[g_{m,n}(\zeta,z) - g_{m,n}(\zeta,\bar{z})\big]\, d\zeta = 0, \quad t \in (0,1] \cup [1,i), \tag{3.13}$$

where $g_{m,n}$ is defined by (2.24) *and $\alpha \in \Delta$ is a fixed constant.*

Proof. Let $\Lambda = \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$ and

$$\begin{aligned} \Upsilon(\zeta,z) = {} & \sum_{(m,n)\in\Lambda} \big[g_{m,n}(\zeta,z) - g_{m,n}(\zeta,\bar{z})\big] + \left(\frac{1}{\zeta + iz - 1 - i} + \frac{1}{\zeta - iz - 1 + i} + \frac{1}{\zeta + z - 2}\right) \\ & - \left(\frac{1}{\zeta + i\bar{z} - 1 - i} + \frac{1}{\zeta - i\bar{z} - 1 + i} + \frac{1}{\zeta + \bar{z} - 2}\right), \end{aligned}$$

then $\Upsilon \in C([i,0]\times\overline{\Delta};\mathbb{C})$ by Lemma 2.3. One has $\bar z \to -t$ as $z \to t \in [i,0]$, $z \in \Delta$, and hence $\Upsilon(\zeta,t)=0$, $t\in[i,0]$, which implies

$$\lim_{z\in\Delta,\, z\to t} \frac{1}{2\pi i}\int_{[i,0]} \rho(\zeta)\Upsilon(\zeta,z)\,d\zeta = 0, \quad t\in[i,0]. \tag{3.14}$$

Since

$$\sum_{m,n}\big[g_{m,n}(\zeta,z)-g_{m,n}(\zeta,\bar z)\big] = \Upsilon(\zeta,z)+\frac{z+\bar z}{(\zeta-z)(\zeta+\bar z)}.$$

It follows that (3.12) is true by Lemma 3.1 and (3.14). On the other hand, by (2.13), $g_{m,n}(\zeta,\bar z)=g_{m,n}(\zeta,z)$, $z\in(0,1]\cup[1,i)$, and hence (3.13) follows by Corollary 2.1. $\square$

Similarly, one has the following lemma.

Lemma 3.3. *If $\rho\in C([1,i];\mathbb{C})$ and $\rho(i)=\rho(1)=0$, then*

$$\lim_{z\in\Delta,\, z\to t} \frac{1}{2\pi i}\int_{[1,i]} \rho(\zeta)\sum_{m,n}\big[g_{m,n}(\zeta,z)-g_{m,n}(\zeta,\bar z)\big]\,d\zeta = \begin{cases}\rho(t), & t\in[1,i],\\ 0, & t\in(i,0]\cup[0,1);\end{cases} \tag{3.15}$$

if $\rho\in C([0,1];\mathbb{C})$ and $\rho(0)=\rho(1)=0$, then

$$\lim_{z\in\Delta,\, z\to t} \frac{1}{2\pi i}\int_0^1 \rho(x)\sum_{m,n}\big[g_{m,n}(x,z)-g_{m,n}(x,\bar z)\big]\,dx = \begin{cases}\rho(t), & t\in[0,1],\\ 0, & t\in(1,i]\cup[i,0),\end{cases} \tag{3.16}$$

where $g_{m,n}$ is defined by (2.24)*.*

Theorem 3.2. *If $\rho\in C(\partial\Delta;\mathbb{R})$ and $\rho(0)=\rho(1)=\rho(i)=0$, then $\{\operatorname{Re} S_\alpha[\rho]\}^+(t)=\rho(t)$, $t\in\partial\Delta$, where S_α is defined by* (3.1)*.*

Proof. Firstly, one has

$$\begin{aligned}\operatorname{Re} F_{m,n}(z) = \frac{1}{2\pi i}\int_{\partial\Delta}\rho(\zeta)\bigg\{&\bigg[g_{m,n}(\zeta,z)-\frac{g_{m,n}(\zeta,\alpha)+g_{m,n}(\zeta,\bar\alpha)}{2}\bigg]d\zeta\\ &-\bigg[\overline{g_{m,n}(\zeta,z)}-\frac{\overline{g_{m,n}(\zeta,\alpha)}+\overline{g_{m,n}(\zeta,\bar\alpha)}}{2}\bigg]d\bar\zeta\bigg\},\quad z\in\Delta,\end{aligned} \tag{3.17}$$

where $F_{m,n}$ is defined by (3.3).

Secondly, we give the representation of $\overline{g_{m,n}(\zeta,z)}\,d\bar\zeta$. From the expression (2.24) of $g_{m,n}$, one easily gets

$$\begin{aligned}\overline{g_{m,n}(\zeta,z)}\,d\bar\zeta = \bigg\{&\frac{1}{\bar\zeta-\bar z-2m+2ni}+\frac{1}{\bar\zeta-i\bar z-(2m+1)+(2n+1)i}\\ &+\frac{1}{\bar\zeta+i\bar z-(2m+1)+(2n-1)i}+\frac{1}{\bar\zeta+\bar z-(2m+2)+2ni}\bigg\}d\bar\zeta.\end{aligned} \tag{3.18}$$

By (2.13), (3.18) can be rewritten into three cases as follows

$$\begin{aligned}\overline{g_{m,n}(\zeta,z)}\,d\bar\zeta = \bigg\{&\frac{1}{\zeta+i\bar z+(2n-1)+(2m-1)i}+\frac{1}{\zeta-\bar z+2n+2mi}\\ &+\frac{1}{\zeta+\bar z+(2n-2)+2mi}+\frac{1}{\zeta-i\bar z+(2n-1)+(2m+1)i}\bigg\}d\zeta,\quad \zeta\in[1,i],\end{aligned} \tag{3.19}$$

$$\begin{aligned}\overline{g_{m,n}(\zeta,z)}\,d\bar\zeta = \bigg\{&\frac{1}{\zeta+\bar z+2m-2ni}+\frac{1}{\zeta+i\bar z+(2m+1)-(2n+1)i}\\ &+\frac{1}{\zeta-i\bar z+(2m+1)-(2n-1)i}+\frac{1}{\zeta-\bar z+(2m+2)-2ni}\bigg\}d\zeta,\quad \zeta\in[i,0]\end{aligned} \tag{3.20}$$

and

$$\overline{g_{m,n}(\zeta,z)}\,d\bar{\zeta} = \left\{\frac{1}{\zeta-\bar{z}-2m+2ni} + \frac{1}{\zeta-i\bar{z}-(2m+1)+(2n+1)i}\right.$$
$$\left. + \frac{1}{\zeta+i\bar{z}-(2m+1)+(2n-1)i} + \frac{1}{\zeta+\bar{z}-(2m+2)+2ni}\right\} d\zeta, \quad \zeta \in [0,1]. \tag{3.21}$$

The three formulas (3.19), (3.20) and (3.21) are equivalent to

$$\overline{g_{m,n}(\zeta,z)}\,d\bar{\zeta} = \begin{cases} h^1_{m,n}(\zeta,\bar{z})\,d\zeta, & \zeta \in [1,i], \\ h^2_{m,n}(\zeta,\bar{z})\,d\zeta, & \zeta \in [i,0], \\ h^3_{m,n}(\zeta,\bar{z})\,d\zeta, & \zeta \in [0,1]. \end{cases} \tag{3.22}$$

Finally, by (3.22), (3.17) can be rewritten in the form

$$\begin{aligned} \operatorname{Re} F_{m,n}(z) = {} & \left\{ \frac{1}{2\pi i} \int_{[1,i]} \rho(\zeta)\big[g_{m,n}(\zeta,z) - h^1_{m,n}(\zeta,\bar{z})\big]\,d\zeta + \frac{1}{2\pi i} \int_{[i,0]} \rho(\zeta)\big[g_{m,n}(\zeta,z) - h^2_{m,n}(\zeta,\bar{z})\big]\,d\zeta \right. \\ & \left. + \frac{1}{2\pi i} \int_0^1 \rho(x)\big[g_{m,n}(x,z) - h^3_{m,n}(x,\bar{z})\big]\,dx \right\} \\ & - \left\{ \frac{1}{2\pi i} \int_{[1,i]} \rho(\zeta)\left[\frac{g_{m,n}(\zeta,\alpha)+g_{m,n}(\zeta,\bar{\alpha})}{2} - \frac{h^1_{m,n}(\zeta,\alpha)+h^1_{m,n}(\zeta,\bar{\alpha})}{2}\right] d\zeta \right. \\ & + \frac{1}{2\pi i} \int_{[i,0]} \rho(\zeta)\left[\frac{g_{m,n}(\zeta,\alpha)+g_{m,n}(\zeta,\bar{\alpha})}{2} - \frac{h^2_{m,n}(\zeta,\alpha)+h^2_{m,n}(\zeta,\bar{\alpha})}{2}\right] d\zeta \\ & \left. + \frac{1}{2\pi i} \int_0^1 \rho(x)\left[\frac{g_{m,n}(x,\alpha)+g_{m,n}(x,\bar{\alpha})}{2} - \frac{h^3_{m,n}(x,\alpha)+h^3_{m,n}(x,\bar{\alpha})}{2}\right] dx \right\}, \quad z \in \Delta, \end{aligned}$$

which leads to

$$\operatorname{Re} S_\alpha[\rho](z) = \frac{1}{2\pi i} \int_{\partial\Delta} \rho(\zeta) \sum_{m,n} \big[g_{m,n}(\zeta,z) - g_{m,n}(\zeta,\bar{z})\big]\,d\zeta, \quad z \in \Delta \tag{3.23}$$

by Corollary 2.1 and Lemma 2.4. By Lemmas 3.2 and 3.3, (3.23) gives

$$\lim_{z\in\Delta,\, z\to t} \operatorname{Re} S_\alpha[\rho](z) = \rho(t), \quad t \in \partial\Delta.$$

This completes the proof. □

Remark 3.1. If $\rho \in C(\partial\Delta;\mathbb{R})$, by (3.2) and (3.23), $S_\alpha[\rho](\alpha) = \operatorname{Re} S_\alpha[\rho](\alpha) \in \mathbb{R}$ when $\alpha \in \Delta$ is a constant.

Lemma 3.4.

$$S_\alpha[1](z) = \frac{1}{\pi i} \int_{\partial\Delta} \sum_{m,n} \left[g_{m,n}(\zeta,z) - \frac{g_{m,n}(\zeta,\alpha)+g_{m,n}(\zeta,\bar{\alpha})}{2}\right] d\zeta = 1, \quad z \in \Delta, \tag{3.24}$$

where the operator S_α is defined by (3.1).

Proof. By (2.34),

$$\frac{1}{\pi i} \int_{\partial\Delta} G_{M,N}(\zeta,z)\,d\zeta = 2, \quad z \in \Delta, \tag{3.25}$$

which in particular implies

$$\frac{1}{2\pi i} \int_{\partial\Delta} G_{M,N}(\zeta,\alpha)\,d\zeta = 1 \tag{3.26}$$

for the fixed constant $\alpha \in \Delta$. In addition,

$$\frac{1}{2\pi i}\int\limits_{\partial\Delta} G_{M,N}(\zeta,\overline{\alpha})\,d\zeta = 0, \quad \alpha \in \Delta. \tag{3.27}$$

Subtracting the sum of (3.26) and (3.27) from (3.25), passing to the limit, the desired conclusion (3.24) is obtained. □

The following result is a generalization of Theorem 3.2.

Theorem 3.3. *If $\rho \in C(\partial\Delta;\mathbb{R})$, then*

$$\big\{\operatorname{Re} S_\alpha[\rho]\big\}^+(t) = \rho(t), \quad t \in \partial\Delta, \tag{3.28}$$

where the operator S_α is defined by (3.1)*.*

Proof. We only need to verify that (3.28) is true at one of corner points, say $t = 0$. Next we prove

$$\big\{\operatorname{Re} S_\alpha[\rho]\big\}^+(0) = \rho(0). \tag{3.29}$$

By Theorem 3.2 and (3.23), one gets

$$\lim_{z\in\Delta,\, z\to 0}\left\{\frac{1}{2\pi i}\int\limits_{[\frac{i}{2},0]\cup[0,\frac{1}{2}]} \rho(\zeta)\sum_{m,n}\big[g_{m,n}(\zeta,z) - g_{m,n}(\zeta,\bar{z})\big]\,d\zeta\right\} = 0 \tag{3.30}$$

for $\rho \in C([\frac{i}{2},0]\cup[0,\frac{1}{2}];\mathbb{R})$ with $\rho(0) = 0$. (3.30) in turn leads to

$$\big\{\operatorname{Re} S_\alpha[\rho]\big\}^+(0) = 0, \quad t \in \partial\Delta \tag{3.31}$$

for $\rho \in C(\partial\Delta;\mathbb{R})$ with $\rho(0) = 0$. If $\rho \in C(\partial\Delta;\mathbb{R})$, by Lemma 3.4,

$$\operatorname{Re} S_\alpha[\rho](z) - \rho(0) = \operatorname{Re} S_\alpha\big[\rho - \rho(0)\big](z) = \operatorname{Re} S_\alpha[\widetilde{\rho}](z)$$

with $\widetilde{\rho}(\zeta) = \rho(\zeta) - \rho(0)$. (3.31) remains true for such $\widetilde{\rho}$, and hence (3.29) is verified. □

4. Properties of Pompeiu-type operators

Suppose that $f \in L_p(\Delta;\mathbb{C})$, $p > 2$, and define the Pompeiu-type operator on Δ

$$A_\alpha[f](z) = -\frac{1}{\pi}\int\limits_{\Delta}\big\{f(\zeta)G_\alpha(\zeta,z) - \overline{f(\zeta)}G_\alpha(\bar{\zeta},z)\big\}\,d\xi\,d\eta, \quad z \in \Delta \tag{4.1}$$

with

$$G_\alpha(\zeta,z) = \sum_{m,n}\big[g_{m,n}(\zeta,z) - g_{m,n}(\zeta,\alpha)\big], \tag{4.2}$$

where $\alpha \in \Delta$ is a fixed constant. Obviously $A_\alpha[f](\alpha) = 0$.

For the classical Pompeiu-type operator [15,19]

$$T[f](z) = -\frac{1}{\pi}\int\limits_{\Delta}\frac{f(\zeta)}{\zeta - z}\,d\xi\,d\eta, \tag{4.3}$$

one has $T[f] \in C(\overline{\Delta};\mathbb{C})$, $\frac{\partial T[f](z)}{\partial\bar{z}} = f(z)$, $z \in \Delta$ and $\frac{\partial T[f](z)}{\partial\bar{z}} = 0$, $z \in \mathbb{C}\setminus\overline{\Delta}$. By Lemma 2.3, one easily gets the following.

Theorem 4.1. *If $f \in L_p(\Delta;\mathbb{C})$, $p > 2$, then $A_\alpha[f] \in C(\overline{\Delta};\mathbb{C})$ and $\frac{\partial A_\alpha[f](z)}{\partial\bar{z}} = f(z)$, $z \in \Delta$, where A_α is defined by* (4.1)*.*

Theorem 4.2. *If $f \in L_p(\Delta;\mathbb{C})$, $p > 2$, then $\{\operatorname{Re} A_\alpha[f]\}^+(t) = 0$, $t \in \partial\Delta$, where A_α is defined by* (4.1)*.*

Proof. By (4.1),

$$\{\operatorname{Re} A_\alpha[f]\}(z) = -\frac{1}{2\pi}\int_\Delta \{f(\zeta)[G_\alpha(\zeta,z) - \overline{G_\alpha(\bar\zeta,z)}] + \overline{f(\zeta)}[\overline{G_\alpha(\zeta,z)} - G_\alpha(\bar\zeta,z)]\}\,d\xi\,d\eta \tag{4.4}$$

with $z \in \overline{\Delta}$. The theorem follows from

$$G_\alpha(\zeta,z) - \overline{G_\alpha(\bar\zeta,z)} = 0, \quad (\zeta,z) \in \Delta \times \partial\Delta. \tag{4.5}$$

Since

$$\begin{aligned}\overline{g_{m,n}(\bar\zeta,z)} &= \frac{1}{\zeta - \bar z - 2m + 2ni} + \frac{1}{\zeta - i\bar z - (2m+1) + (2n+1)i} \\ &\quad + \frac{1}{\zeta + i\bar z - (2m+1) + (2n-1)i} + \frac{1}{\zeta + \bar z - (2m+2) + 2ni},\end{aligned} \tag{4.6}$$

implies that

$$\overline{g_{m,n}(\bar\zeta,z)} = g_{m,-n}(\zeta,\bar z), \quad z \in [0,1] \tag{4.7}$$

one has

$$G_\alpha(\zeta,z) - \overline{G_\alpha(\bar\zeta,z)} = 0, \quad z \in [0,1]. \tag{4.8}$$

When $z \in [1,i]$, then $z = 1 - t + ti$, $t \in [0,1]$ by the parametrization (1.1), and hence (4.6) gives

$$\begin{aligned}\overline{g_{m,n}(\bar\zeta,z)} &= \frac{1}{\zeta + t + ti - (2m+1) + 2ni} + \frac{1}{\zeta - t + ti - (2m+1) + 2ni} \\ &\quad + \frac{1}{\zeta + t - ti - (2m+1) + 2ni} + \frac{1}{\zeta - t - ti - (2m+1) + 2ni}.\end{aligned} \tag{4.9}$$

On the other hand,

$$\begin{aligned}g_{m,n}(\zeta,z) &= \frac{1}{\zeta + t - ti - (2m+1) - 2ni} + \frac{1}{\zeta - t - ti - (2m+1) - 2ni} \\ &\quad + \frac{1}{\zeta + t + ti - (2m+1) - 2ni} + \frac{1}{\zeta - t + ti - (2m+1) - 2ni}.\end{aligned} \tag{4.10}$$

By (4.9) and (4.10), $\overline{g_{m,n}(\bar\zeta,z)} = g_{m,-n}(\zeta,z)$, $z \in [1,i]$, which leads to

$$G_\alpha(\zeta,z) - \overline{G_\alpha(\bar\zeta,z)} = 0, \quad z \in [1,i]. \tag{4.11}$$

Similarly, substituting $z = i(1-t)$, $t \in [0,1]$ into $\overline{g_{m,n}(\bar\zeta,z)}$ and $g_{m,n}(\zeta,z)$, respectively, one has

$$G_\alpha(\zeta,z) - \overline{G_\alpha(\bar\zeta,z)} = 0, \quad z \in [i,0]. \tag{4.12}$$

Then (4.8), (4.11) and (4.12) are equivalent to (4.5). This completes the proof. □

5. Schwarz-type BVP of Cauchy–Riemann equation

Now we consider the following Schwarz-type problem: Find a solution w of the nonhomogeneous Cauchy–Riemann equation

$$\partial_{\bar z} w(z) = f(z), \quad z \in \Delta \tag{5.1}$$

satisfying the Schwarz-type boundary condition

$$[\operatorname{Re} w]^+(t) = \rho(t), \quad t \in \partial\Delta, \tag{5.2}$$

where $f \in L_p(\Delta;\mathbb{C})$, $p > 2$ and the boundary data $\rho \in C(\partial\Delta;\mathbb{R})$.

Theorem 5.1. *The Schwarz-type BVP* (5.1) *with* (5.2) *is solvable and its solution can be expressed as*

$$w(z) = S_\alpha[\rho](z) + A_\alpha[f](z) + ic, \quad z \in \Delta, \tag{5.3}$$

where S_α and A_α are defined by (3.1) *and* (4.1)*, respectively, and $c = \operatorname{Im} w(\alpha)$ with $\alpha \in \Delta$.*

Proof. By Theorems 3.1 and 4.1, the function $\widetilde{w}(z) = S_\alpha[\rho](z) + A_\alpha[f](z)$ is a solution of the nonhomogeneous Cauchy–Riemann equation (5.1), and also satisfies the boundary condition (5.2) by Theorems 3.3 and 4.2. On the other hand, $\widetilde{w}(\alpha) = S_\alpha[\rho](\alpha) \in \mathbb{R}$ by Remark 3.1 since $A_\alpha[f](\alpha) = 0$. If w is an arbitrary solution of the Schwarz-type BVP (5.1) with (5.2), then $w - \widetilde{w}$ solves the homogeneous Schwarz-type problem of analytic functions on Δ, which implies $w(z) - \widetilde{w}(z) = ic$, $z \in \overline{\Delta}$ with $c \in \mathbb{R}$ by Theorem 2.1. Hence $w(z) = S_\alpha[\rho](z) + A_\alpha[f](z) + ic$ and $w(\alpha) = S_\alpha[\rho](\alpha) + ic$, which leads to $c = \operatorname{Im} w(\alpha)$. This completes the proof of the theorem. □

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